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Numerical Evaluation of the Azimuthally Dependent Albedo Problem in Slab Geometry

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The azimuthal dependence of the neutron transport equation in a plane geometry is treated by a Fourier decomposition. The solution to the slab albedo problem is obtained by expansion of the azimuthal dependence of the angular flux into an infinite Fourier cosine series resulting in a finite set of uncoupled, azimuthally symmetric slab albedo problems. The Fourier coefficients of the angular flux are shown to satisfy a modified transport equation which can readily be solved numerically with existing codes. Numerical results are presented for a one-speed example problem, and a comparison of a ten-group calculation with an experimental transmission spectrum is given.

I. INTRODUCTION

The solution to the slab albedo problem of transport theory consists of the angular density of particles in a finite, plane-parallel medium surrounded by a vacuum, with a monoenergetic, monoangular beam of particles uniformly incident upon one face of the slab. In general, the angular flux will depend upon two angular variables, a polar angle and an azimuthal angle. The assumption of azimuthal symmetry is commonly made to reduce the dimensionality of the problem.

In this study, the azimuthally dependent slab albedo problem is investigated. By an expansion of the azimuthal dependence in a Fourier series, a finite set of uncoupled, azimuthally symmetric slab albedo problems result, which can be solved by standard numerical techniques.

The general transport problem with azimuthal dependence has been studied extensively only for photon transport, primarily by the Monte Carlo and orders-of-

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scattering techniques [1, 2]. The orders-of-scattering technique has been applied to the slab albedo problem in neutron transport by Thiesing [3], but his limitation to second order scatterings neglects a significant contribution to the angular flux. The Fourier decomposition in the azimuth has been applied to particle transport by several authors [4–7], using several different analytical techniques. The results, however, are highly formalistic and not amenable to direct numerical computations.

The derivation of the Fourier azimuthal decomposition is summarized in Section II. In Section III the computational considerations are discussed and the numerical results are presented in Section IV.

II. Fourier Decomposition

The one-speed transport equation in the plane geometry coordinate system of Fig. 1 may be written as

$$\mu \frac{\partial \psi(x, \mu, \phi)}{\partial x} + \psi(x, \mu, \phi) = c \int_{\Omega'} d\Omega' f(\mathbf{\Omega} \cdot \mathbf{\Omega}') \psi(x, \mu, \phi), \tag{1}$$

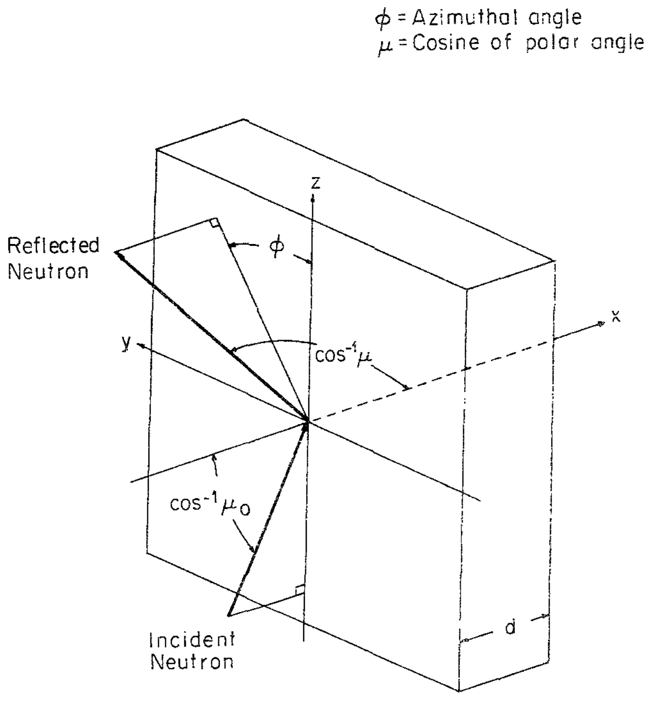

FIG. 1. Coordinate system for slab geometry.

where $\psi(x, \mu, \phi)$ is the total angular flux as a function of both the polar angle, $\cos^{-1} \mu$, and the azimuthal angle ϕ. Distance is measured in mean free paths, and c is the mean number of secondary neutrons per collision. The scattering function, $f(\mathbf{\Omega} \cdot \mathbf{\Omega}')$, is assumed rotationally invariant. For the slab albedo problem with a uniformly distributed slant source, the boundary conditions are

$$\psi(0, \mu, \phi) = \delta(\mu - \mu_0)\, \delta(\phi - \phi_0), \qquad \mu, \mu_0 > 0, \tag{2a}$$

and

$$\psi(d, \mu, \phi) = 0, \qquad \mu < 0, \tag{2b}$$

where d is the slab thickness and δ is the Dirac delta function. It is noted that this basic albedo solution is in the form of a "Green's function" so that superposition will generate the solution for a distributed input. The scattering function is approximated by an $N + 1$ term Legendre polynomial series expansion [5], which upon use of the "addition" theorem [8] becomes

$$f(\mathbf{\Omega} \cdot \mathbf{\Omega}') = \frac{1}{2\pi} \sum_{k=0}^{N} c_k \left[\tilde{P}_k(\mu)\, \tilde{P}_k(\mu') + 2 \sum_{j=1}^{k} \tilde{P}_k^j(\mu)\, \tilde{P}_k^j(\mu') \cos j(\phi - \phi') \right], \tag{3}$$

where c_k are the known expansion coefficients. The tilde indicates use of the normalized Legendre functions [8] which are employed for notational convenience.

The azimuthal dependence of the angular flux is expanded in a Fourier cosine series

$$\psi(x, \mu, \phi) = \psi_0(x, \mu) + \sum_{n=1}^{\infty} \psi_n(x, \mu) \cos n(\phi - \phi_0). \tag{4}$$

The $\psi_n(x, \mu)$ are the unknown Fourier azimuthal expansion coefficients. Since the flux is symmetric about the ϕ_0 plane and the cosine functions are complete for even functions, the series of Eq. (4) will converge to $\psi(x, \mu, \phi)$, provided the angular flux is integrable over the ϕ-domain.

Insertion of the expansions, Eq. (3) and Eq. (4), into Eq. (1) and integration over the azimuthal angles yields

$$\mu \frac{\partial \psi_0(x, \mu)}{\partial x} + \psi_0(x, \mu) = c \sum_{k=0}^{N} c_k \tilde{P}_k(\mu) \int_{-1}^{1} d\mu'\, \tilde{P}_k(\mu')\, \psi_0(x, \mu'). \tag{5a}$$

This equation is the standard azimuthally symmetric transport equation used in the one-angle approximation. The boundary conditions are

$$\psi_0(0, \mu) = \frac{1}{2\pi}\, \delta(\mu - \mu_0), \qquad \mu, \mu_0 > 0, \tag{5b}$$

and

$$\psi_0(d, \mu) = 0, \qquad \mu < 0. \tag{5c}$$

Repeating the above procedure, with a weighting factor of cos $m(\phi - \phi_0)$ before integration over all azimuthal angles, yields a "modified" transport equation for the higher azimuthal modes, viz.,

$$\mu \frac{\partial \psi_m(x, \mu)}{\partial x} + \psi_m(x, \mu) = c \sum_{k=m}^{N} c_k \tilde{P}_k^m(\mu) \int_{-1}^{1} d\mu' \tilde{P}_k^m(\mu') \psi_m(x, \mu'), \quad m = 1, 2, \ldots, N, \tag{6a}$$

and

$$\mu \frac{\partial \psi_m(x, \mu)}{\partial x} + \psi_m(x, \mu) = 0, \qquad m > N. \tag{6b}$$

The appropriate boundary conditions from Eqs. (2) and (4) are

$$\psi_m(0, \mu) = \frac{1}{\pi} \delta(\mu - \mu_0), \qquad \mu, \mu_0 > 0, \tag{6c}$$

and

$$\psi_m(d, \mu) = 0, \qquad \mu < 0, \quad m = 1, 2, \ldots, \infty. \tag{6d}$$

The solutions for the higher order modes ($m > N$) of Eq. (6b) can be obtained analytically. These solutions, when inserted into the Fourier expansion, Eq. (4), yield

$$\psi(x, \mu, \phi) = \psi_0(x, \mu) + \sum_{m=1}^{N} \psi_m(x, \mu) \cos m(\phi - \phi_0) + \sum_{m=N+1}^{\infty} \frac{1}{\pi} \exp(-x/\mu_0) \delta(\mu - \mu_0) \cos m(\phi - \phi_0). \tag{7}$$

The infinite sum in Eq. (7) is slowly convergent and undesirable from computational considerations. The elimination of this infinite sum is accomplished by separation of the collided and uncollided transmission in the lower Fourier modes, viz.,

$$\psi_0(x, \mu) = \hat{\psi}_0(x, \mu) + \frac{1}{2\pi} \exp(-x/\mu_0) \delta(\mu - \mu_0) \tag{8a}$$

and

$$\psi_m(x, \mu) = \hat{\psi}_m(x, \mu) + \frac{1}{\pi} \exp(-x/\mu_0) \delta(\mu - \mu_0), \quad m = 1, 2, \ldots, N, \tag{8b}$$

where $\hat{\psi}$ represents the diffuse or collided contribution to the azimuthal modes. If the uncollided part of the lower Fourier modes is absorbed into the infinite sum of Eq. (7), the Fourier cosine expansion for the delta function $\delta(\phi - \phi_0)$, is immediately recognized. The Fourier expansion for the angular flux then attains the final form

$$\psi(x, \mu, \phi) = \hat{\psi}_0(x, \mu) + \sum_{m=1}^{N} \hat{\psi}_m(x, \mu) \cos m(\phi - \phi_0) + \delta(\mu - \mu_0)\, \delta(\phi - \phi_0) \exp(-x/\mu_0). \tag{9}$$

Thus the azimuthally dependent slab albedo problem is reduced by a Fourier cosine series expansion in azimuth to an $N + 1$ set of uncoupled, azimuthally symmetric slab albedo problems. It is noted from Eq. (9) that any calculation of the angular flux based only on the azimuthally symmetric component (i.e., the one-angle approximation) will neglect a contribution from the higher modes that may be significant. However, if some integral quantity is of interest, such as the emergent currents, the integral of the higher modes will vanish, so that knowledge of the azimuthally symmetric component is sufficient.

Although the above derivation is restricted to one-speed transport, the extension to multigroup transport is direct.

III. Computational Considerations

Since the modified transport equation, Eq. (6a), is similar in form to the standard azimuthally symmetric transport equation, many existing numerical techniques [9–11] are applicable directly to the calculation of the higher Fourier modes. In particular, the ANISN discrete ordinates code [9], without major modification, is used for the numerical results shown in the next section. Physical arguments require the total angular flux, $\psi(x, \mu, \phi)$, to be positive throughout the slab, as also must be the azimuthally symmetric component, $\psi_0(x, \mu)$. However, no such assertions can be made about the higher Fourier modes. Consequently, the negative flux fixup procedure in ANISN must be bypassed in the computation of these higher modes.

The final form of the Fourier cosine expansion, Eq. (9), has the uncollided transmission separated from the diffuse contribution. The ANISN code computes only the total angular flux, including the uncollided transmission. The collided component can, in principle, be found by subtraction of the analytically-known uncollided fraction from the ANISN-computed total angular flux. In practice, this procedure is satisfactory only for the lower Fourier modes. For the higher Fourier modes, however, the uncollided transmission becomes a large fraction of the total angular flux. Consequently, the above subtraction procedure becomes quite susceptive to numerical inaccuracies. To avoid this difficulty, two dummy quadrature angles (angles with zero weights) are placed very close to either side of the input source angle. The diffuse angular flux for the Fourier modes is then found by simple linear interpolation between the angular fluxes at the two dummy angles.

IV. Numerical Results

To demonstrate the behavior of the azimuthal dependence in the slab albedo problem, the results of two one-speed calculations are presented in this section. The following theoretical scattering function is used in the calculation [12].

$$f^{N\pm}(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}') = (N+1)(1\pm\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')^N/2^N, \qquad \text{with} \qquad N = 10. \tag{10}$$

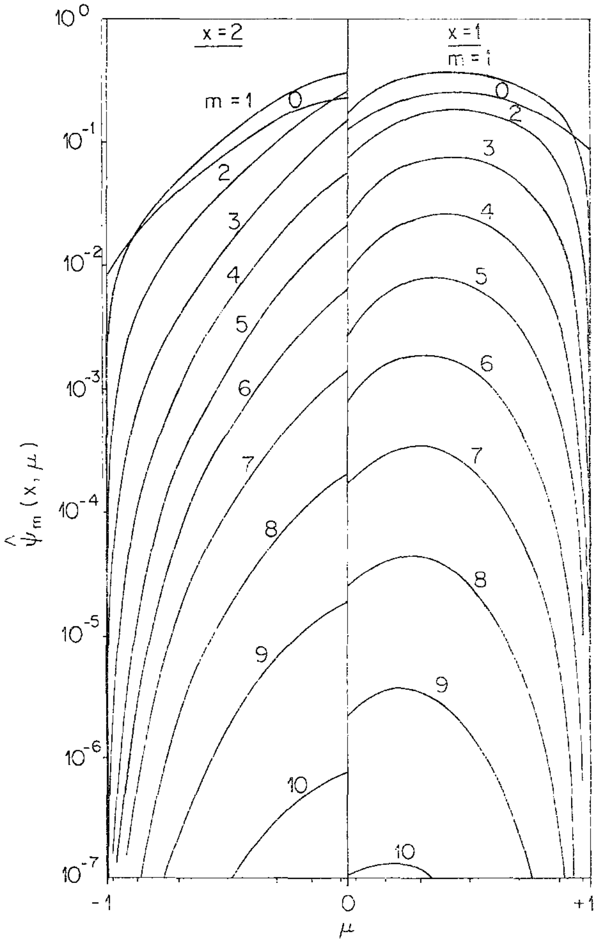

FIG. 2. Tenth-order forward scattering fourier coefficients.

This function is a strongly peaked forward (backward) scattering function using the positive (negative) sign. The parameters of the model problem are $d = 1.0$, $c = 0.95$, $\mu_0 = 0.5$, and $\phi_0 = 0°$.

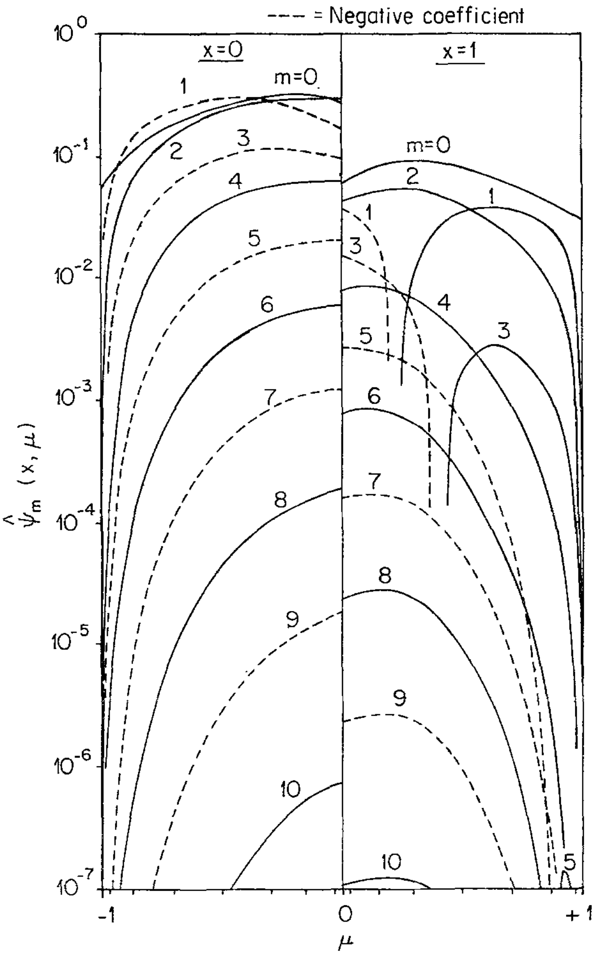

FIG. 3. Tenth-order backward scattering fourier coefficients.

Choice of the DP_7 angular quadrature results in coincidence of a quadrature ordinate and the source input angle. Dummy quadrature angles are placed at $\mu = 0.4999$ and 0.5001 for the evaluation of the collided angular flux at the source input angle by linear interpolation. Calculation of the higher Fourier modes by Eq. (6a) requires the zeroth moment of the scattering function, c_0, to be zero.

Under such conditions, the inner iterations of ANISN fail to converge. Approximation of this moment by 10^{-30} avoids the difficulty.

Since the ANISN code does not possess an external boundary source option, the slab albedo problem cannot be modelled directly. To simulate this boundary source at $x = 0$, the first spatial mesh is chosen as 10^{-7} cm. thick with zero macroscopic cross sections and a source emitting neutrons at $\mu_0 = 0.5$. The remaining spatial intervals are chosen to have a thickness of 0.1 mean free paths.

The Fourier azimuthal modes for the forward and backward scattering function are shown in Figs. 2 and 3, respectively. All of the higher modes ($m > 1$) vanish at $\mu = \pm 1$ as is expected from the flux symmetry requirements. For the forward scattering function all the modes are positive; however for the backward scattering function the odd modes may be negative over part or all of the μ range.

The azimuthal dependence of the emergent angular fluxes for the forward and backward scattering functions are plotted in Figs. 4 and 5, respectively. In all cases,

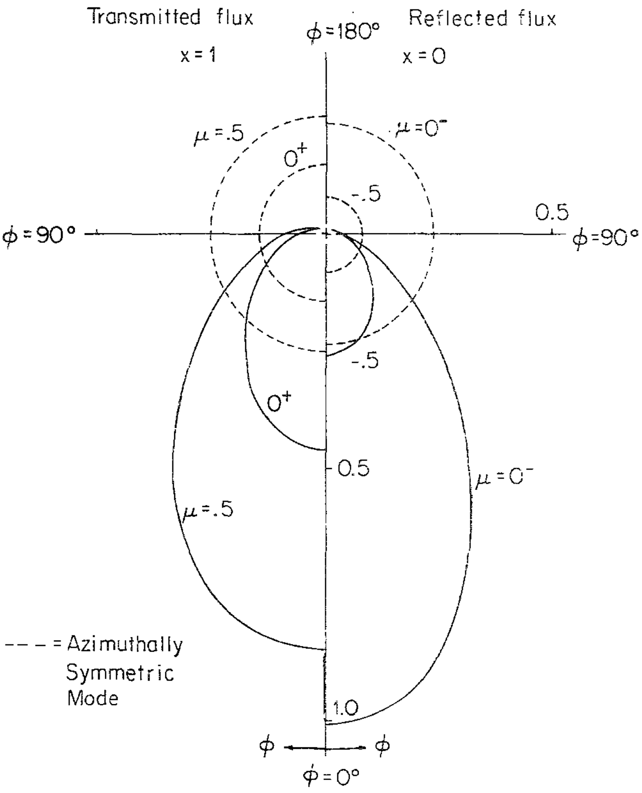

FIG. 4. Polar plot of the tenth-order forward scattering angular flux.

the azimuthally symmetric component is seen to be a poor approximation for the azimuthal dependence. The relative azimuthally dependent flux at the mid point in the slab, $\psi(0.5, \mu, \phi)$, is represented as a three dimensional surface in Fig. 6 for both the forward and backward scattering functions. The contour lines of constant ϕ are spaced 10° apart, and since the flux is symmetric about the $\phi = 0° - 180°$ plane, only the first 180° in ϕ is plotted. The contour lines of constant μ are the DP_7 quadrature ordinates.

A more practical application of the azimuthally dependent calculation is furnished by a multigroup example. The transmitted fast neutron energy distribution through a 4.62-cm steel slab with an incident pencil beam source at $\theta_0 = 60°$ and $\phi_0 = 0°$ is reported by Thiesing [3]. This transmission spectrum, shown in Fig. 7, is for a slab illuminated by a pencil beam source and a detector that

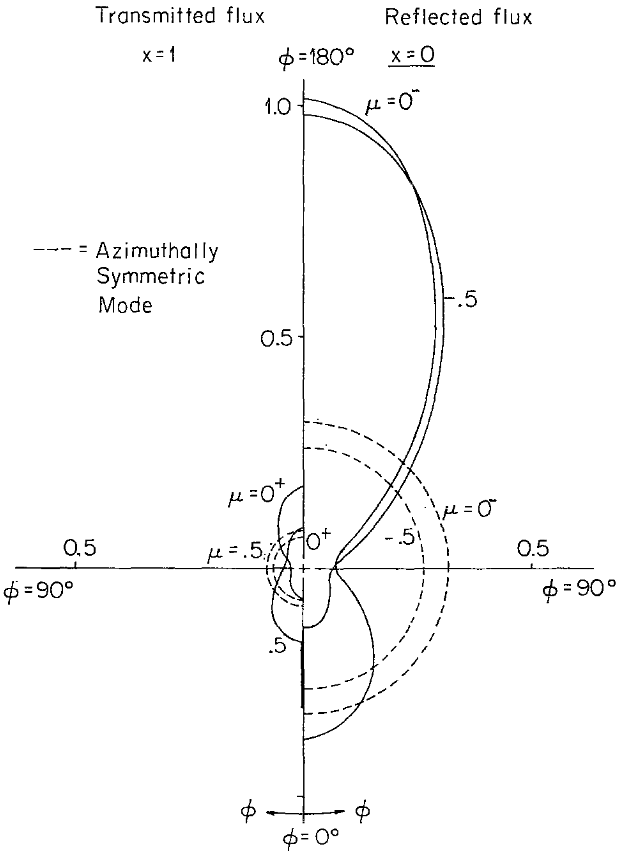

FIG. 5. Polar plot of the tenth-order backward scattering angular flux.

integrates the transmitted angular flux over the polar and azimuthal angle. For thin slabs and large source-to-detector distances, this may be approximated by a slab illuminated with a uniform slant source and a collimated detector directed at the point at which the pencil beam strikes the slab.

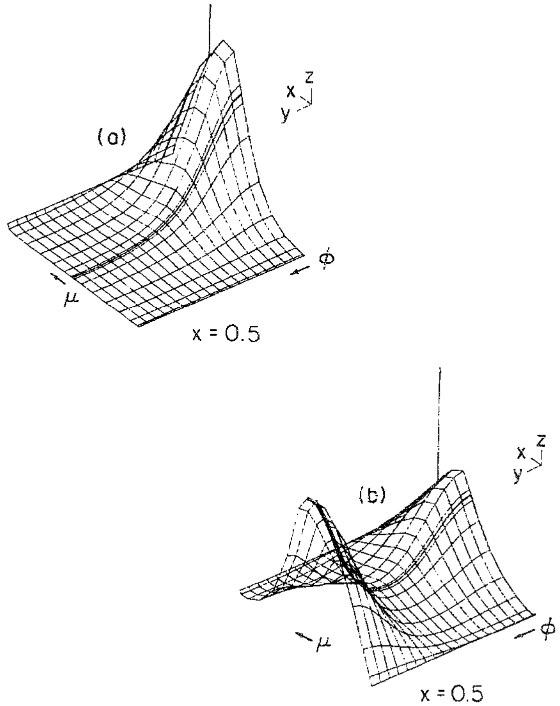

FIG. 6. Relative angular flux at slab midpoint for forward and backward scattering: (a) forward scattering, (b) backward scattering.

The ANISN calculation of the Fourier coefficients for this transmission problem are performed using the DP_7 angular quadrature, 20 spatial intervals and the first ten groups of the DLC/2C cross section set [13] for steel. The incident fast neutron source spectrum is that reported by Thiesing [3]. Modification of ANISN to eliminate the coarse mesh rebalance was necessary to obtain convergence of the inner iterations.

The azimuthal expansion coefficients from ANISN display many negative values, even in the azimuthally symmetric component, primarily in the direction of reflected neutrons. The existence of many negative scattering probabilities at the backward scattering angles in the DLC/2C set is reported by Thiesing [3], and this difficiency in the DLC/2C set produces the unrealistic results, particularly for the higher Fourier modes, in the reflected fluxes. The transmitted fluxes, however, are not nearly as sensitive to the cross section errors.

The transmitted energy spectrum computed by the Fourier decomposition technique is compared with the experimental transmitted spectrum in Fig. 7, normalized to the experimental value at 7.8 MeV.

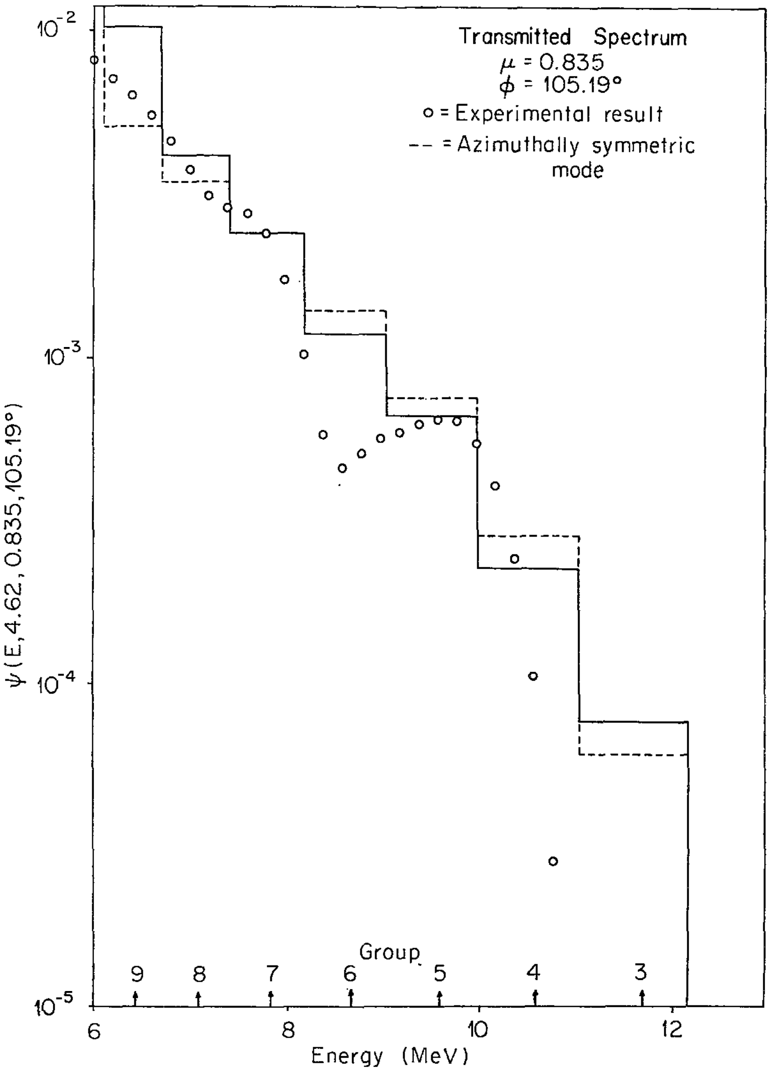

FIG. 7. Ten group iron slab transmission spectrum.

SUMMARY

The analysis in this paper indicates that azimuthally dependent albedo problem in slab geometry can be treated in a very direct manner by use of a Fourier decomposition of the azimuthal dependence. It is shown that the ANISN transport code may be applied directly to the solution of this problem. The extension to azimuthally dependent problems other than the albedo problem may be treated in a similar manner.

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